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IMPERFECTION SENSITIVITY OF OPTIMIZED, LAMINATED COMPOSITE SHELLS: A PHYSICAL APPROACH

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Abstract—Optimization of the axial buckling load of composite, cylindrical shells through a judicious choice of laminate configuration is often associated with increased imperfection sensitivity. Current approaches of combining postbuckling theory with an optimization program demand highly sophisticated analytical and computational methods, yet are insufficient to provide a rational theme that can be used to derive general design guidelines. The present paper is an attempt to explore the subject matter via a different avenue, such that various nonlinear effects may be understood in physical terms which require relatively little in the way of advanced mathematics and computation. The paper proposes to study the problem using a simple, but intuitively appealing, reduced stiffness analysis of cylinder buckling which recognizes the physical characteristics present in advanced postbuckling and uses them in an equivalent linear, eigenvalue analysis.

This investigation highlights the specific relationship between laminate stiffness parameters, efficiency of buckling resistance and imperfection sensitivity in postbuckling deformation. It is observed that the criteria for optimality and reduced imperfection sensitivity are often opposed to each other. The reduced buckling load appears to be a useful indicator for evaluating qualitatively the relative imperfection sensitivity of various nearly optimal laminated shell designs which would be of great interest to designers. Another interesting feature is the analytical study in terms of bounded generic orthotropic constants which furnishes a general theme on the issue. A comprehensive discussion on the theoretical foundation of the reduced stiffness approach and other similar approximate methods is provided. It has been shown throughout this paper that the proposed physical approach successfully and consistently explains most of the observations reported in the literature which were based on nonlinear postbuckling analyses.

NOTATION

d	$1-\nu_{12}\nu_{21}$
D^*	generalized rigidity ratio
E_1	longitudinal elastic modulus
E_2	transverse elastic modulus
G	shear modulus
h	single lamina thickness
j	circumferential wavenumber
L	length of shell
N_x	compressive stress resultant
N_c	classical buckling load
$\bar{N}_c$	nondimensional classical buckling load
N_r	reduced buckling load
$\bar{N}_r$	nondimensional reduced buckling load
N_s	equivalent isotropic shell buckling load
Q_{11}	$= E_1/d$
Q_{22}	$= E_2/d$
Q_{12}	$= \nu_{12}Q_{22}$
Q_{66}	$= G$
R	radius of shell
t	thickness of shell
Z	$= L^2/Rt$
α	principal rigidity ratio
λ	$= j\pi R/L$
ε	generalized Poisson's ratio
ρ	$= N_r/N_c$
η	$= N_c/N_s$
ξ	$= \mu/t$
μ	axisymmetric imperfection amplitude
θ	fiber orientation in a lamina
ν_{12}	major Poisson's ratio
ν_{21}	minor Poisson's ratio.

INTRODUCTION

The buckling behaviour of circular cylindrical, laminated, composite shells under axial compression is an important design consideration, as a significant feature of composite technology is tailoring capability and optimization of performance through the additional degrees of freedom thereby introduced, viz. fiber orientation and distribution in each lamina, laminae thickness, stacking sequence and material properties. It is observed that optimization with respect to the linear buckling load introduces significant changes in postbuckling behaviour which, in most cases, may prove highly undesirable from a practical performance point of view because of the large increase in imperfection sensitivity. This has prompted interest in studying the effects of fiber orientation, stacking sequence and lamina thicknesses on the imperfection sensitivity of axially compressed laminated composite shells.

Current approaches generally apply Koiter's asymptotic peturbation method or nonlinear imperfect equilibrium analysis in conjunction with an optimization program for estimating imperfection sensitivity of the generated optimal laminate designs. These combined postbuckling and optimization exercises rely heavily on intensive, but physically remote, analytical and computational methods which neither help in undestanding the physics of the structural behaviour nor offer scope for incorporation into the design-synthesis process. Naturally, the complexity of the problem and a large number of design variables make a parametric study a prohibitively expensive exercise. In the literature, numerical results for very select examples are reported which are insufficient to provide general design recommendations.

The present paper is an attempt to explore the subject matter via a different avenue such that various nonlinear effects may be understood in physical terms which require relatively little in the way of advanced mathematics and computation. Here, the aim is to highlight the application of certain striking physical ideas to the complex issue of imperfection sensitivity, which are susceptible to study in a relatively simple and unsophisticated manner yet led to sound hypotheses. With this philosophical background, this paper proposes to study the present problem using a simple, but intuitively appealing, reduced stiffness analysis of cylinder buckling first introduced by Croll and Batista (1981) for isotropic shells.

The method is based on the concept that modal coupling and imperfections, in the postbuckling range, will result in the erosion of the initial stabilization provided by the quadratic circumferential membrane energy component, V_c, the result of Poisson bulging in the fundamental state. This has led to the simple idea that a lower limit to the buckling strength can be provided by a reduced critical load which can be obtained by ignoring the specific energy component, V_c, in the linear buckling analysis. The present study attempts to establish the reduced buckling load as a useful measure of imperfection sensitivity in the optimization process. Emphasis is placed on developing a consistent physical model which can be used to derive general design guidelines.

This paper is organized in the following manner. Firstly, a brief review of the relevant literature and a summary of the philosophy of the reduced stiffness method is presented. An energy method for classical and reduced stiffness buckling analysis of axially compressed cylinders is extended to laminated orthotropic composite shells. The results of the proposed physical approach are compared with those obtained by nonlinear analysis (Simitses and Sheinman, 1982) and its theoretical foundation is discussed in greater depth in view of recent analytical/numerical findings. The imperfection sensitivities of a number of optimal laminated shell designs, available in the literature, are examined using the proposed criterion. Finally, the analysis is presented in a generic form which establishes a set of physical parameters affecting imperfection sensitivity and efficiency of laminates and, therefore, offers a rational theme for use in design.

It has been shown throughout the paper that, based on nonlinear postbuckling analysis, most of the observations reported in the literature can be explained via a reduced stiffness buckling model contained in a remarkable contribution by Croll and Batista (1981).

BACKGROUND LITERATURE

The stability of thin, circular cylindrical shells has received considerable attention over past decades. It is possible, but somewhat impractical, to provide an overview of the large amount of theoretical and experimental results available on this topic and thus attention is focussed strictly on the title paper. The interested reader is referred to an excellent review by Simitses (1986) covering recent developments in this area.

The two general approaches for imperfection sensitivity analysis of composite shells, namely, Koiter's asymptotic perturbation method and full nonlinear equilibrium analysis of the imperfect shell, are briefly reviewed. Some of the optimization studies are summarized and attempts at approximate analysis of the problem are enumerated.

Using Koiter's (1945) method, Tennyson and Hansen (1982) presented a detailed study on optimum design for buckling of laminated composite cylinders including their postbuckling response and imperfection sensitivity. They reported that varying the stacking sequence can more than double the buckling strength. However, the increased load capability was also asssociated with increased imperfection sensitivity which was measured by computing Koiter's b-coefficient for the particular design of interest. Some of the results are more interesting than surprising. For example, two graphite/epoxy shells made of [90/0/0/90] and [0/90/90/0] laminates, respectively, have similar linear buckling loads but substantially different postbuckling characteristics; shell 1, [90/0/0/90], exhibiting stable postbuckling behaviour ($b > 0$) while shell 2, [0/90/90/0], is unstable ($b < 0$). Sun (1987) considered a typical four-ply laminate, $[90, \theta, -\theta, 90]$, and studied the effects of fiber orientation, θ, on buckling strength and imperfection sensitivity as measured by the b-coefficient in the presence of various imperfection magnitudes. His results cannot be easily rationalized. Semenyuk and Zhukova (1987) also studied various laminated shell designs using Koiter's general method and observed a relationship between the b-coefficient and the ratios of the longitudinal and transverse elastic moduli to the shear modulus, the larger these two ratios, the lower the imperfection sensitivity. This will be examined in the following sections. These observations could not be generalized with confidence because of a limited number of examples. Naturally, the complexity of the problem and a large number of design variables make parametric study prohibitively expensive and impractical.

Sheinman and Simitses (1977) presented a complex but more accurate numerical solution scheme for nonlinear stability analysis based on the von Karman–Donnell nonlinear kinematic relations. The computational procedure for obtaining the critical limit point load of imperfect shells employed a Fourier series type of separated solutions; through the Galerkin procedure the field equations were reduced to a system of ordinary differential equations and subsequently solved by a finite difference scheme. This method was extended to study the post limit point response of imperfect isotropic shells (Simitses and Sheinman, 1982). Using this procedure, Simitses *et al.* (1985) established the imperfection sensitivity of composite shells through plots of critical (limit) loads versus imperfection amplitude. The larger the drop in critical load value with increasing amplitude, the greater was the sensitivity. Simitses and has associates have studied the problem in great depth and provided useful numerical results for comparative purposes. The interesting features of their results will be discussed later.

Sun and Hansen (1988) combined Koiter's general method with an optimization program and reported that this introduced significant changes in the imperfection sensitivity of shells. This suggests the existence of multiple optimal laminate configurations with varying degrees of imperfection sensitivity and post-buckling response. This notion will be examined thoroughly in the rest of this paper. The following two optimization studies are selected for our purposes. Kobayashi *et al.* (1982) presented optimal designs of laminates composed of three types of layers, axial ($\theta = 0$), circumferential ($\theta = 90$) and helical ($0 < \theta < 90$). The optimization was carried out with respect to the number, thickness and stacking sequence of the three basic layers. It was found that many laminate designs exist, corresponding to an optimal buckling load. Nshanian and Pappas (1982) applied a mathematical programming technique to obtain the optimal ply angle and through-thickness distribution in symmetric laminates. In cases of axially loaded cylinders, the existence of multiple optimal solutions was reported.

The increasing reliance on numerical methods has motivated many researchers to focus on fundamental conceptual notions of postbuckling behaviour and to develop simplified design methods. Here, a cursory review of some such attempts is presented. At present, a discussion of the reduced stiffness method (Croll and Batista, 1981) is avoided as it will be dealt with, in detail, later in the paper. Calladine and Robinson (1978) strongly advocated a simplified treatment of shell buckling problems. Taking a clue from the formal analogy between Koiter's imperfection sensitivity formula and that of Ayrton and Perry (1886) for a simple column, they explained important physical aspects of imperfection sensitivity. The notion they proposed was the steady growth of imperfection amplitude as the compressive load on the shell increases, bringing progressive changes in the stress resultants throughout the shell; collapse occurs when the largest stress reaches a value equal to the classical buckling stress, an idea similar to the growth of bending stress in Ayrton and Perry column. This simple approach allowed them to write load-imperfection relations qualitatively similar to that of Koiter's (1945); quantitatively, the agreement between the two was extremely poor. Walker and Sridharan (1980) argued that, since the curvature, and, in particular, w/R term in the circumferential strain increases a shell's resistance to initial buckling compared with the corresponding flat plate, the same curvature term is also responsible for its marked imperfection sensitivity. It was, therefore, suggested that a reliable lower bound to buckling strength of a shell is its strength as a flat plate, with curvature playing only the role of biasing the shell to buckle in a mode corresponding to its lowest critical stress. Thus, the neutral equilibrium or lower bound stress is equal to the buckling stress of the prismatic flat plate structure whose side is equal to the developed halfwave length of the buckled cylinder.

It may be slightly out of context to mention the appproximate buckling analysis of pressurized cylinders, yet it highlights an interesting feature. Croll (1975) explained that unstable postcritical behaviour is the result of loss of membrane stiffness in the presence of imperfections and proposed a quasi-inextensional linear energy analysis ignoring the contribution from membrane energy in order to obtain a lower bound buckling load. Wittek (1982) followed a similar idea in a finite element analysis and studied the relation between imperfection magnitude and loss of the membrane energy component. This approach could not be extended to axially loaded cylinders since, in that case, the membrane and bending energy provide roughly equal contributions to the resistance to critical deformation. Thus, neglecting the membrane strain energy would lead to a lower bound prediction of 50% of the classical buckling load for the whole range of problem variables, which does not conform with experimental observations.

REDUCED STIFFNESS MODEL OF SHELL BUCKLING

It is very clear that current approaches in this area rely heavily on complex analytical and computational methods which do not help in understanding the physics of structural behaviour. Croll and Batista (1981) made a significant contribution by introducing a reduced stiffness analysis for cylinder buckling which, in essence, was an application of Donnell's (1934) rationale for coupling modes in the postbuckling regime. The approach is based on the very simple notion that it is difficult to close something that is not already there at the outset. Thus, an analogy is drawn in which the loss of carrying capacity of a shell in the elastic postbuckling regime represents the loss of certain components of its initial stiffness which should be capable of explanation in terms of initial problem parameters, i.e. geometry, deformation and associated energy terms.

The basic philosophy of the reduced stiffness method is summarized very briefly in symbolic form and details can be found elsewhere (Croll and Batista, 1981). The quadratic, incremental potential energy, V_2, of a given shell subjected to axial compression may be written as,

$$V_2 = U_c - V_w \tag{1}$$

where U_c is internal strain energy and V_w is the work done by stresses in the fundamental state

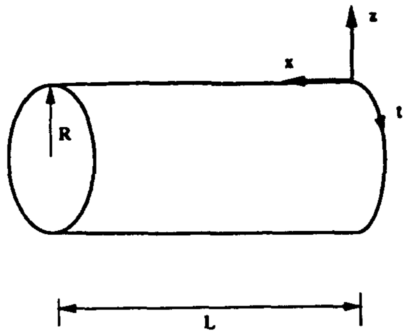

Fig. 1. Coordinate axes.

on quadratic membrane strains. V_w can be decomposed into its axial and circumferential components, V_x and V_t, respectively, thus

$$V_w = V_x - V_t. \quad (2)$$

It is to be noted that V_t, known as the quadratic circumferential membrane energy, is a result of Poisson expansion (bulging) occurring in the fundamental state. Now, substituting for V_w from eqn (2) into eqn (1)

$$V_2 = U_c - [V_x - V_t] \quad (3)$$

and regrouping terms leads to

$$V_2 = [U_c + V_t] - V_x. \quad (4)$$

Thus, it is the term V_t which contributes to initial stabilization. Reiterating Donnell's (1934) arguments and using a physical model, Croll and Batista (1981) demonstrated that, in the postbuckling regime, coupling of the periodic critical deformation mode with an axisymmetric mode of half the axial wavelength will result in erosion of the initial stabilization provided by V_t. This led to the simple idea that a lower limit to the shell postcritical stiffness would be provided by a **reduced** incremental quadratic potential energy,

$$(V_2)_{red} = U_c - V_x, \quad (5)$$

obtained by eliminating V_t; the buckling load thus obtained is referred to as the reduced buckling load. This reduced stiffness model is found to predict closely the mode triggering buckling in a large number of experiments and to provide reliable lower bounds.

ANALYTICAL FORMULATION

The Rayleigh–Ritz energy method presented buy Croll and Batista (1981) is extended to the buckling analysis of laminated, composite circular cylindrical shell of length L, radius R, and thickness t, simply supported at the ends (Fig. 1).

The membrane strain vector, e, is expressed as the sum of the prebuckling strain, E, and incremental linear and quadratic membrane strain vectors, e' and e'', respectively, while $\bar{N}$, N, n' and n'' are correspondingly associated stress vectors. Here subscripts "x" and "t" are used to indicate axial and circumferential coordinate axes, respectively. Thus, the components of the strain vector are

$$\begin{aligned} e_x &= E_x + e'_x + e''_x \\ e_t &= E_t + e'_t + e''_t \\ e_{xt} &= E_{xt} + e'_{xt} + e''_{xt}. \end{aligned} \quad (6)$$

Similarly,

$$\begin{aligned}\bar{N}_x &= N_x + n'_x n''_x \\ \bar{N}_t &= N_t + n'_t + n''_t \\ \bar{N}_{xt} &= N_{xt} + n'_{xt} + n''_{xt}.\end{aligned} \tag{7}$$

Using Donnell approximations, the strain–displacement relations may be written as

$$e'_x = \frac{\partial u}{\partial x},\quad e'_t = \frac{1}{R}\left(\frac{\partial v}{\partial \phi} + w\right),\quad e'_{xt} = \frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \phi},\quad e''_x = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2,\quad e''_t = \frac{1}{2R^2}\left(\frac{\partial w}{\partial \phi}\right)^2. \tag{8}$$

Here, u, v and w denote displacements in the axial (x), circumferential (t) and radial (z) directions, as shown in Fig. 1. The curvature terms are defined as

$$k_x = -\frac{\partial^2 w}{\partial x^2},\quad k_t = -\frac{\partial^2 w}{R^2 \partial \phi^2},\quad k_{xt} = -\frac{2\partial^2 w}{R\,\partial x\,\partial \phi}. \tag{9}$$

According to the general stability theory (Croll and Walker, 1972), the quadratic components, V_2, of the total potential energy control the stability of equilibrium of the fundamental path and lead to an eigenvalue problem yielding the critical stable state of the shell. The incremental strain energy, U_e, may be written as (Croll and Batista, 1981)

$$U_e = \frac{1}{2}\int_0^l \int_0^{2\pi} (n'_x e'_x + n'_t e'_t + n'_{xt} e'_{xt}) r \,\mathrm{d}\phi\,\mathrm{d}x + \frac{1}{2}\int_0^l \int_0^{2\pi} (m_x k_x + m_t k_t + m_{xt} k_{xt}) r\,\mathrm{d}\phi\,\mathrm{d}x, \tag{10}$$

in contrast to classical analysis where the work done by the axial load is given as

$$V_w = \int_0^l \int_0^{2\pi} N_x e''_x r\,\mathrm{d}\phi\,\mathrm{d}x. \tag{11}$$

Croll and Batista (1981) restructured this term as $V_w = V_x - V_t$, where

$$V_x = \frac{1}{2}\int_0^l \int_0^{2\pi} (N_x e''_x + n''_x E_x) r\,\mathrm{d}\phi\,\mathrm{d}x$$

and

$$V_t = \frac{1}{2}\int_0^l \int_0^{2\pi} (n''_t E_t) r\,\mathrm{d}\phi\,\mathrm{d}t. \tag{12}$$

Constitutive relation

Classical lamination theory, based on Kirchhoff's hypothesis, gives force ($\bar{N}$) and moment (m) resultants by the following constitutive relations (Jones, 1975):

$$\begin{Bmatrix}\bar{N}\\ m\end{Bmatrix} = \begin{bmatrix}A_{ij} & B_{ij}\\ B_{ij} & D_{ij}\end{bmatrix}\begin{Bmatrix}\bar{e}\\ k\end{Bmatrix}\quad i,j = 1,\ldots,3 \tag{13}$$

where A_{ij}, B_{ij} and D_{ij} ($i,j = 1,\ldots,3$) represent the stretching, coupling and bending stiffness matrices of a laminate and are defined as follows

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-t/2}^{t/2} Q_{ij}(1, z, z^2)\,\mathrm{d}z\quad (i,j = 1,\ldots,3). \tag{14}$$

Here, $Q_{11} = E_1/d$, $Q_{22} = E_2/d$, $Q_{12} = \nu_{12}Q_{22}$, $Q_{33} = G$ and $d = 1-\nu_{12}\nu_{21}$. The details of the derivation can be found elsewhere (Jones, 1975). It is assumed that the in-plane stiffness matrix, A_{ij}, has orthotropic form, i.e. $A_{13} = A_{23} = 0$.

The linear membrane theory is usd to derive the prebuckling strain vector, E:

$$E_x = N_x C_1, \quad E_t = N_x C_2 \quad \text{and} \quad E_{xt} = 0 \tag{15}$$

where

$$C_1 = \frac{A_{22}}{\Delta}, \quad C_2 = \frac{A_{12}}{\Delta} \quad \text{and} \quad \Delta = A_{11}A_{22}-A_{12}^2. \tag{16}$$

Displacement functions

The classical, simply supported boundary conditions, i.e. @ $x = 0, l$ $w = m_x = v = 0$ and $\bar{N}_x$ = const, are assumed which are satisfied by the following displacement functions

$$u = U\cos\frac{j\pi x}{L}\sin i\phi$$

$$v = V\sin\frac{j\pi x}{L}\cos i\phi$$

$$w = W\sin\frac{j\pi x}{L}\sin i\phi. \tag{17}$$

These boundary conditions can be exactly satisifed by symmetric laminates with $B_{ij} = 0$ and cross-ply laminates with B_{11} and B_{22} the only non-zero terms in the B_{ij} submatrix. In the case of an antisymmetric angle-ply, with B_{13} and B_{23} the only non-zero terms in the B_{ij} submatrix, the force boundary conditions are violated.

Substituting the assumed strain displacement and constitutive relations and appropriate derivatives of the displacement functions into eqns (10) and (12), energy expressions are derived explicitly and are listed in Appendix A. Stationarity of the quadratic, incremental potential energy with respect to the kinematically admissible displacements (u, v, w) gives the following conditions

$$\frac{\partial V_2}{\partial U} = \frac{\partial V_2}{\partial V} = \frac{\partial V_2}{\partial W} = 0 \tag{18}$$

which lead to a set of linear homogeneous algebraic equations of the form:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{bmatrix} \begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = 0 \tag{19}$$

where

$$K_{11} = A_{11}\lambda^2 + A_{33}i^2$$

$$K_{12} = (A_{12}+A_{33})i\lambda$$

$$K_{13} = -A_{12}\lambda - \frac{B_{11}\lambda^3}{R}$$

$$K_{22} = A_{22}i^2 + A_{33}\lambda^2$$

$$K_{23} = = A_{22}i - \frac{B_{22}i^3}{R}$$

$$K_{33} = K'_{33} + N_x C_x - N_x C_t \tag{20}$$

where

$$K'_{33} = A_{22} + \frac{1}{R^2}[D_{11}\lambda^4 + 2(D_{12}+2D_{33})i^2\lambda^2 + D_{22}i^4]$$

$$C_x = \lambda^2 + (A_{11}\lambda^2 + A_{12}i^2)C_1$$

$$C_t = (A_{12}\lambda^2 + A_{22}i^2)C_2$$

$$\lambda = \frac{j\pi R}{L}.$$

The buckling criterion is the existence of nontrivial solutions of eqn (19) which is dictated by the vanishing determinant of $[K]$. Thus,

$$\det[K] = 0. \tag{21}$$

Equation (21) generates axial buckling load spectra for various axial and circumferential wave numbers; the minimum of which is the classical, linear buckling load of the shell.

The reduced critical load is obtained by ignoring the energy term V_t in the quadratic potential energy which, in turn, is equivalent to ignoring term $N_x C_t$ in the expression for K_{33} in eqn (20) and subsequently invoking condition (21).

BOUNDS ON THE REDUCED CRITICAL LOAD

The effects of shell length, L, on the reduced critical load, N_r, were studied; for this purpose, the lower and upper bounds corresponding to $L = \infty$ and $L = 0$, respectively, were derived. It is to be noted that the classical buckling load, N_c, is independent of length.

The buckling load depends on internal strain energy and the load potential such that

$$N_c \propto \frac{U_c}{V_x - V_t} \quad \text{and} \quad N_r \propto \frac{U_c}{V_x}. \tag{22}$$

If we assume that the critical wave numbers (i, j) corresponding to the classical and reduced loads are the same, so that the total energy in the two cases also remains the same, the ratio of the two loads can be written as

$$\frac{N_r}{N_c} = \frac{V_x - V_t}{V_x} = 1 - \frac{V_t}{V_x}. \tag{23}$$

Using the appropriate expressions for V_x and V_t from Appendix A and substituting for E_x and E_t from eqn (15) gives

$$\frac{V_t}{V_x} = \frac{(A_{12}\lambda^2 + A_{22}i^2)C_2}{\lambda^2(1 + A_{11}C_1) + A_{12}i^2C_1}. \tag{24}$$

As $L \to \infty$, $\lambda = j\pi R/L \to 0$ and therefore eqn (24) results in

$$\frac{V_t}{V_x} = \frac{A_{22}i^2C_2}{A_{12}i^2C_1}. \tag{25}$$

Substituting for C_1 and C_2 from (16) and using eqn (23), one can find that, in the limit

$$L \to \infty, \quad \frac{V_t}{V_x} \approx 1 \quad \text{and} \quad \frac{N_r}{N_c} \approx 0. \tag{26}$$

Of course, $L = \infty$ is a hypothetical case and our basic assumption, as stated above, may not be true, in general, but it clearly illustrates the point that the reduced buckling load would decrease with increasing length of slenderness, similar to that of an Euler column. In the derivation of the reduced buckling load, N_r, the V_t energy term is neglected which, in effect, implies that the prebuckling circumferential strain, E_t, and hence the Poisson's effect, is ignored. It provides a clue to the observed relationship.

The other bound can be obtained by setting $L = 0$ in eqn (24). Thus

$$\frac{V_t}{V_x} = \frac{A_{12}C_2}{1+A_{11}C_1} \tag{27}$$

and the limiting ratio of the reduced to classical loads becomes

$$\frac{N_r}{N_c} = 1 - \frac{A_{12}C_2}{1+A_{11}C_1} \tag{28}$$

which will always have a finite value on account of Poisson's effect. For an isotropic shell, $A_{11} = A_{22} = E/(1-\nu^2)$ and $A_{12} = \nu A_{11}$, which upon substituting in eqn (28) results in

$$\frac{N_r}{N_c} = 1 - \frac{\nu^2}{2-\nu^2}. \tag{29}$$

For $\nu = 0.3$, $N_r/N_c = 0.953$.

Efficiency of buckling resistance of laminates

For a given orthotropic composite material, the maximum buckling load attainable is (Tennyson, 1987)

$$N_s = \frac{E_s t^2}{c_s R}, \quad c_s = \sqrt{3(1-\nu_s^2)} \tag{30}$$

and is referred to as the buckling resistance of an equivalent isotropic laminate which is independent of lamination sequence. Thus, the ratio of the classical buckling load to the equivalent isotropic buckling load is defined as the efficiency, η, of the laminate in buckling, i.e.

$$\eta = \frac{N_c}{N_s}. \tag{31}$$

The equivalent isotropic properties, elastic modulus, E_s, Poisson's ratio, ν_s, and shear modulus, G_s, are defined in terms of the invariants of the composite material as follows:

$$\nu_s = \frac{U_4}{U_1}, \quad E_s = (1-\nu_s^2)U_1, \quad G_s = U_5$$

where

$$U_1 = \tfrac{1}{8}(3Q_{11}+3Q_{22}+2Q_{12}+4Q_{33})$$
$$U_4 = \tfrac{1}{8}(Q_{11}+Q_{22}+6Q_{12}-4Q_{33})$$
$$U_5 = \tfrac{1}{8}(Q_{11}+Q_{22}-2Q_{12}+4Q_{33}). \tag{32}$$

Table 1. Comparison of reduced stiffness and nonlinear analyses isotropic shells. $E = 10.5 \times 10^6$ psi, $\nu = 0.3$, $R = 4$ in.

Shell no.	R/t	L/R	λ^l (Simitses and Sheinman, 1982)	λ^m	ρ Red. stiff. model
1	250	1	0.258 (8)	0.105 (6)	0.396 (10)
2	250	3	0.595 (5)	0.113 (5)	0.168 (6)
3	250	5	0.719 (4)	0.117 (4)	0.108 (5)
4	250	8	0.873 (3)	0.326 (3)	0.069 (3)
5	250	10	0.875 (3)	0.392 (3)	0.062 (3)
6	1000	1	0.446 (13)	0.022 (9)	0.237 (15)
7	500	1	0.344 (10)	0.076 (8)	0.309 (12)
8	250	1	0.248 (8)	0.106 (6)	0.396 (10)
9	80	1	0.157 (5)	0.151 (5)	0.557 (7)

Numbers in brackets denote circumferential wavenumber.

NUMERICAL RESULTS AND DISCUSSION

In this section, the basic proposal that the ratio of reduced to classical buckling load, ρ, can be treated as a measure of imperfection sensitivity is examined with the aid of numerical examples.

Comparison with nonlinear analysis

The effectiveness of the reduced stiffness model is studied by comparing its lower bound predictions, ρ, with results of nonlinear analysis of imperfect isotropic shells (Simitses and Sheinman, 1982). The lower bound, ρ, limit point, λ^l, and minimum postlimit point loads, λ^m, the last two values as obtained by Simitses and Sheinman (1982), are presented in Table 1 for a variety of shell geometries. The results of nonlinear analysis correspond to $\xi = 1$ and an asymmetric imperfection of amplitude 0.1ξ.

The lower bound load and corresponding wavenumber are in remarkably close agreement with λ^m for shells 2 and 3 and close to λ^l for shell 7. The lower bound prediction is nonconservative for shells 1, 9 and overly conservative for shells 4, 5. Shells 1–5 indicate a consistent drop in ρ with increasing length, and thus follow the predicted trend. The implication of this relationship is in contrast to Simitses and Sheinman's conclusion, that is, since λ^l and λ^m increase with increasing length, longer shells are less imperfection sensitive. In all cases, the buckling mode associated with the reduced buckling load is unique and always consists of a single axial halfwave ($j = 1$) and a number of circumferential waves often close to those corresponding to classical buckling. This mode is often observed to trigger buckling in experiments (Croll and Batista, 1981). At this point, it is worth mentioning some potentially serious drawbacks of nonlinear numerical analyses (Croll, 1975), namely numerical instability, convergence to some misleading complementary equilibrium paths in the vicinity of limit points, and nonconservative results due to incorrect sign of the critical imperfection in the case of asymmetric bifurcation. The dependence of ρ on imperfections is not explicit but is rather based on physical arguments and, therefore, testing its validity by comparison with the results of nonlinear analysis for one particular set of imperfections may not be justified.

Example problems

Three representative problems on buckling of laminated cylinders are considered. The geometry and material properties are described in Tables 2 and 3. Firstly, a symmetric

Table 2. Shell geometries and laminate configurations

Example	Lamination	Material	R (in.)	t (in.)	L/R	N_x (lb in.$^{-1}$)
1	$(90, -\theta, -\theta, 90)$	boron–epoxy	7.50	0.0212	1, 2, 5	417.5
2	$(90, \theta, \theta, 90)$	graphite–epoxy	2.82	0.0171	4	490.3
3	$(\theta, 0, \theta)$	glass–epoxy	5.94	0.0360	4	630.9

Table 3. Material properties

Material	E_1 (10^6 psi)	E_2 (10^6 psi)	G (10^6 psi)	ν_{12}	D^*	α	ε	η	ρ	$\bar{N}_c$ (10^{-2})
Boron e.	30	2.7	0.65	0.21	0.207	0.30	0.304	0.29	0.97	0.135
Graphite e.	20.5	1.4	0.59	0.26	0.287	0.26	0.236	0.31	0.96	0.168
Glass e.	7.5	3.5	1.25	0.35	0.699	0.68	0.342	0.82	0.29	0.265

angle-ply laminate is analyzed and variations of η and ρ with respect to θ are shown in Fig. 2. It confirms an earlier observation (Simitses *et al.*, 1985) that $\theta = 45$ is not a good choice as it exhibits poor efficiency and significantly less ρ. A steep drop in reduced buckling load in the vicinity of optima, $\theta \approx 20, 70$, is observed.

The second example which involves $(90, \theta, \theta, 90)$ graphite–epoxy laminates is motivated by Sun's results (1987) on a similar clamped laminated shell. The ρ vs θ plot in Fig. 3 has qualitative similarities with Koiter's b-coefficient vs θ plots for higher imperfections, $\xi \approx 0.2$, 0.3 (Sun, 1987, Fig. 10). For example, the b-coefficient has the largest negative value and thus higher imperfection sensitivity for $\theta \approx 60$–65°. In Fig. 3, ρ assumes its minimum value in the same range. Also, the b-coefficient for $(90, 0, 0, 90)^\circ$ is higher than $(90, 90, 90, 90)$ and so is the value of ρ.

The last example considers $(\theta, 0, \theta)$ glass–epoxy laminates, which are similar to those of Tennyson *et al.* (1971). The variation of ρ with θ is smoother when compared with earlier examples.

Imperfection sensitivity of multiple optimal laminates

In earlier examples, the thicknesses of the constituent laminae are assumed constant and only fiber orientation is varied. In a more general optimization exercise, not only the fiber orientation in a lamina but also its thickness can be considered as variable. Nashanian and Pappas (1982) and Kobayashi *et al.* (1982) have reported useful results in this area. The most interesting feature of their work is that for a given total thickness, more than one optimal laminate can be found such that their linear buckling loads are almost equal. As a designer, it would be of great interest to examine their imperfection sensitivity and select the safer configuration.

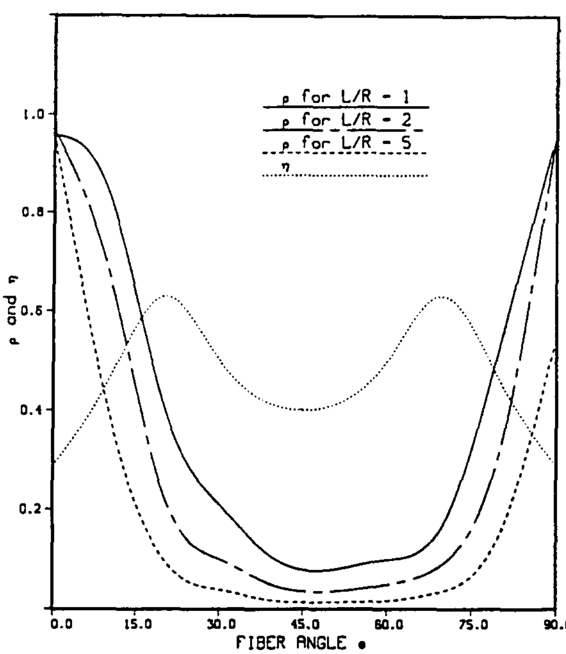

Fig. 2. Buckling of $(\theta, -\theta, -\theta, \theta)$ laminated shells.

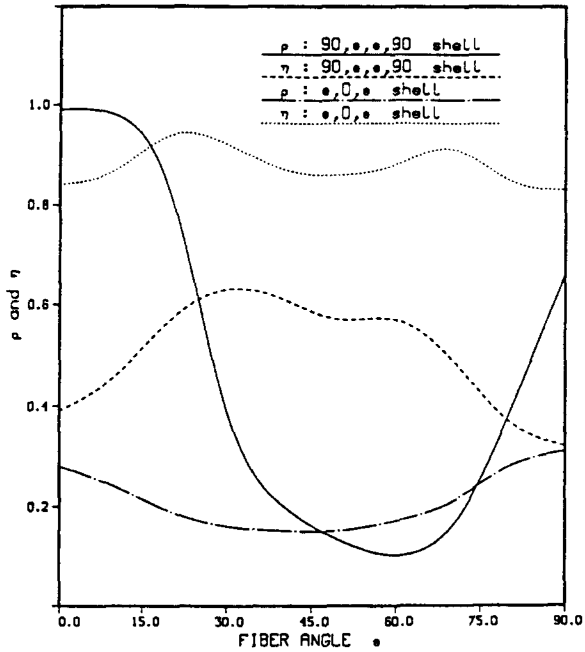

Fig. 3. Reduced buckling load and efficiency of laminated shells.

Table 4 contains results on optimal symmetric laminates of typical designs, which involve 0, 90 and θ° layers. The optimal value, θ, and thickness distribution among the three layers have been obtained for a constant total thickness, $t = 1$ mm and $R = 200$ mm, $L = 600$ mm (Kobayashi *et al.*, 1982). The material properties are $E_1 = 1.5 \times 10^4$ kgf mm^{-2}, $E_2 = 9.62 \times 10^2$ kgf mm^{-2}, $G = 5.16 \times 10^2$ kgf mm^{-2} and $\nu = 0.32$. The pairs of laminates with equal critical load, such as 1–2, 3–4, 5–6 etc. exhibit very different reduced critical loads. The bending stiffness matrices, D_{ij}, for such a pair of laminates are found to have some distinct features. In both laminates, D_{12} and D_{33} elements are identical and within a pair

$$\left(\frac{D_{11}}{D_{22}}\right)_{\text{laminate 1}} \times \left(\frac{D_{11}}{D_{22}}\right)_{\text{laminate 2}} = 1. \tag{33}$$

Within the pair, the laminate with $D_{22} > D_{11}$ always has the higher value of ρ. For example, in group 5–6, Table 4, laminate 5 has $D_{22} > D_{11}$ ($D_{11} > D_{22}$ for laminate 6) and the value of ρ is more than double that of 6. These findings suggest that the laminate with higher hoop bending stiffness (D_{22}) is less imperfection sensitive. This fact, which had earlier been

Table 4. Buckling analysis of laminated cylinders

Shell no.	Lamination	(N_c) (Kgf mm^{-1})	ρ	η
1	[90 (0.2), 32 (0.3)]	11.99	0.35	0.69
2	[0 (0.2), 58 (0.3)]	12.07	0.17	0.69
3	[90 (0.15), 29 (0.35)]	13	0.37	0.75
4	[0 (0.15), 61 (0.35)]	13.26	0.15	0.76
5	[90 (0.1), 25 (0.4)]	13.75	0.39	0.79
6	[0 (0.1), 65 (0.4)]	13.81	0.15	0.79
7	[25 (0.4), 90 (0.1)]	11.23	0.17	0.64
8	[65 (0.4), 0 (0.1)]	10.37	0.28	0.59

All laminates are symmetric about their midplane and only the half-laminate geometry is shown. The fractions in brackets denote the thickness of the respective lamina.

Table 5. Reduced stiffness analysis of laminated cylinder

Shell no.	Lamination	N_c ($kgf\,mm^{-1}$)	ρ	η
1	[60, 0, −60]	9.04	0.19	0.92
2	[30, 90, −30]	9.27	0.15	0.94
3	[0, 55, 0]	7.16	0.32	0.73
4	[90, 35, 90]	7.16	0.23	0.73
5	[90, 0, 45]	7.25	0.47	0.74
6	[30, 0, 90]	7.01	0.25	0.71
7	[60, 90, 0]	6.90	0.50	0.70
8	[0, 90, 45]	7.25	0.27	0.74
9	[55, 0, 0, −55]	15.78	0.26	0.90
10	[35, 90, 90, −35]	16.00	0.16	0.92
11	[55, 0, −55, 90]	15.47	0.18	0.89
12	[45, 0, −45, 90]	14.57	0.15	0.84

All laminates are symmetric about their midplane and only the upper halves of the laminates are shown. Each lamina is 0.125 mm thick.

pointed out by Tennyson and Hansen (1982) using Koiter's theory, is now supplemented with a more physical explanation.

Table 5 contains more examples of laminates with almost equal buckling load and different imperfection sensitivity. In this group, laminates with constant lamina thickness, $h = 0.125$ mm, $L = 600$ mm and $R = 200$ mm are considered. Examples 1–2 and 9–12 are exactly as reported by Kobayashi *et al.* (1982) while, in examples 3–8, the laminates are assumed symmetric rather than antisymmetric as reported in the original paper. The reason is that the assumed displacement functions in our simple energy method do not satisfy boundary conditions for antisymmetric laminates due to extension–twist coupling, i.e. nonzero B_{13} and B_{23} terms. Our conjecture about the relation between the relative magnitude of hoop bending stiffness and imperfection sensitivity is, in general, satisfied by laminate pairs with nearly equal buckling loads. Laminates 3–4 are an exception.

Table 6 summarizes results on optimal symmetric laminates (Nashanian and Pappas, 1982) of constant total thickness, $t = 0.2$ in. and shell radius, $R = 6$ in. for two sets of the parameter, $Z = 60$ and 1500. The material properties are $E_1 = 30 \times 10^6$ psi, $E_2 = 0.75 \times 10^6$ psi, $G = 0.375 \times 10^6$ psi and $\nu = 0.25$. For the first set, $Z = 60$, among laminates 2 and 3 which have almost equal buckling loads, laminate 3 is preferable due to a higher value of ρ. Laminates 3 and 6, corresponding to higher values of ρ, have $D_{11} > D_{22}$ and thus do not follow our earlier conjecture. At the same time, it is found that their values of element A_{12} are higher than of laminates 2 and 5, respectively. Thus, not only the hoop bending stiffness but also the Poisson in-plane stiffness term, A_{12}, has a marked influence on the imperfection sensitivity.

Table 6. Reduced stiffness analysis of optimized cylinders

Shell no.	Lamination	N_c (10^5 lb in.$^{-1}$)	ρ	η
	$Z = 60$			
1	[22 (0.1)]	0.165	0.72	0.38
2	[36 (0.069), 89 (0.031)]	0.403	0.39	0.94
3	[47 (0.04), 18 (0.024), 84 (0.036)]	0.411	0.53	0.96
	$Z = 1500$			
4	[12 (0.1)]	0.169	0.45	0.39
5	[38 (0.07), 90 (0.03)]	0.279	0.11	0.65
6	[42 (0.023), 0 (0.047), 73 (0.03)]	0.341	0.30	0.79

All laminates are symmetric about their midplane and only the upper half is shown. The fractions in brackets are the respective lamina thicknesses.

Remarks

The basic ideas behind the proposed approach are discussed in the light of some of the analytical results recently reported in the literature.

The reduced stiffness model provides a lower bound load beyond which, in the advanced postbuckling regime, the shell is expected to have stable equilibrium. The shell can buckle at such a load level due to the presence of larger imperfections and stiffness loss in modal coupling. In fact, Pedersen's (1974) findings on advanced postbuckling behaviour of imperfect shells support such a line of thinking. He reported that the presence of relatively larger axisymmetric imperfections decreases the bifurcation stress significantly, but, at the same time, stabilizes advanced postbuckling behaviour in the sense that shells could support larger loads than the buckling load before collapse occurs. This held true even in cases where initial postbuckling behaviour, in Koiter's sense, indicated an unstable bifurcation. Sun's results (1987) on a clamped $(90, \theta, -\theta, 90)$ laminated cylinder follow a similar trend. He found extreme sensitivity to smaller imperfections and more stable postbuckling, marked by a positive b-coefficient, combined with large imperfections for a wide range of fiber orientations, θ. Simitses and Sheinman (1982) reported for shell 6 in Table 1 that with increase in imperfection amplitude, λ^{m} increases slowly while λ^{l} decreases drastically and both values approach each other. Eventually, for $\xi = 4$, a very large imperfection, both become almost identical, i.e. $\lambda^{l} \approx \lambda^{m} \approx 0.08$.

Recently, Geier and Rohwer (1989) studied the comparative postbuckling behaviour of an optimized shell panel, as reported by Zimmermann (1982), against a reference using a nonlinear finite element program. Optimized and reference laminates consisted of 16 layers with fiber orientations of $[(\pm 26.1)_s(\pm 55.8)_s]_s$ and $[90_{16}]$, respectively. The load-deflection behaviour of the optimized design (Figs 13 and 14 in Geier and Rohwer, 1989) was marked by a severe load drop of almost 50%, indicating extreme imperfection sensitivity when compared to the reference design, in spite of the fact that the linear buckling load of the optimized panel was almost twice the reference value. Reduced stiffness analyses of cylindrical shells of the above-mentioned optimized and reference configurations, respectively, provide the following results: buckling loads, $N_c = 35.48$ and 17.36 N mm^{-1}, laminate efficiency, $\eta = 0.96$ and 0.47, and reduced critical load ratio, $\rho = 0.48$ and 0.79. Material constants were taken from Zimmermann (1982). The prediction of imperfection sensitivity of the optimized design ($\rho = 0.48$) using the reduced stiffness model is in close agreement with accurate, nonlinear finite element analysis. In the case of the optimized laminate, significantly less hoop bending stiffness, D_{22}, and exceedingly large in-plane Poisson's stiffness, A_{12}, compared to the reference design is noteworthy, reinforcing the notion regarding the relationship between imperfection sensitivity and laminate stiffness parameters. The buckling mode corresponding to the reduced buckling load, most often, consists of one long axial wave, a fact that is further verified by Geier and Rohwer (1989), who reported that advanced postbuckling is generally characterized by one deep buckle.

Becker *et al.* (1982) analyzed the nonlinear behaviour of two 8-ply cylindrical panels of configurations, $(90, \pm 45, 0)_s$ and $(90, 0)_{2s}$, respectively, using a STAGS-C code. For these two laminates, they found nondimensional buckling load of 45.4 and 33.3, and a ratio of nonlinear collapse to buckling load as 0.71 and 0.88, respectively. Thus, the first laminated panel, with higher buckling load, seems to be less reliable in the nonlinear range. The reduced stiffness analysis of two shells of identical laminates, material and geometry, estimates the postbuckling load-carrying capacity in terms of ρ as 0.74 and 0.98 and the laminate efficiency, η, as 0.65 and 0.48, respectively, which is in reasonably close agreement with accurate analysis. Interestingly, the hoop bending stiffness, D_{22}, is identical for both laminates but the in-plane stiffness, A_{12}, for the first is quite large compared to the second, which seems to make it more imperfection sensitive.

The coupling of modes further complicates the issue. In contrast to classical approaches, Hunt *et al.* (1986) explored the problem using a mathematical concept of symmetry breaking. The recognized harmonic buckling modes for axially loaded cylinders are, by nature, symmetric, but combinations of these may lead to asymmetrical deformation, the Yoshimura pattern. The effects of the symmetries, which appear in the buckling modes but not in the final deformed shape and thus not in the underlying governing differential equations, have revealed various interesting features of this complex problem. They reported that interaction between any chequerboard mode and its axisymmetric counterpart, which have well separated critical loads on the fundamental path, is responsible for symmetry breaking effects and associated severe imperfection sensitivity. This reinforces the point that multi-mode interaction is an inherent and inescapable part of cylinder buckling and one cannot rely on remoteness of a potentially interacting mode on the fundamental path as insurance against severe destabilizing effects. Thus, no single mode alone represents a good Rayleigh–Ritz approximation, since none accommodate the fundamental underlying asymmetry. Therefore, b-coefficients computed on the basis of a single mode approximation (Tennyson and Hansen, 1982; Sun, 1987) cannot reliably be used for predicting imperfection sensitivity, on the ground that the presence of complicated but pronounced modal coupling effects may result in a behaviour very far from single mode analysis. Hunt *et al.* (1986) complimented the reduced stiffness model in their words, *Croll (1981) associates the destabilization and subsequent restabilization of cubic and quartic energy terms with a particular quadratic term* (V_i) *and thus gets results from a linear eigenvalue analysis that compare well with experiment.*

The discussion highlights the complexities of postbuckling and imperfection sensitivity which certainly cannot be encapsulated by the reduced stiffness model in a precise manner. At the same time, this discussion substantiates the ideas behind this physical model of imperfection sensitivity by quoting similar observations from various numerical/analytical studies. In many cases, the proposed simple analysis is found to predict the behaviour mechanism remarkably well. For example, shell 3, in Table 1, undergoes snap-through instability at $\lambda^1 = 0.719$ ($n = 4$) and modal coupling brings down its strength to $\rho = 0.108$ ($n = 5$) which is also confirmed by computer analysis, $\lambda^m = 0.117$ ($n = 4$). Therefore, it is maintained that the reduced critical load can serve as a useful indicator in guiding the design-optimization process. Observing the effects of fiber orientation, laminate configuration and material properties on various energy terms, particularly the one which will be eroded in postbuckling, provides a better understanding of the problem and justifies the application of the proposed approximate method.

GENERIC REDUCED STIFFNESS ANALYSIS

In earlier examples, analyses are limited to specific material constants, viz. graphite–epoxy, glass–epoxy, etc., which do not provide a comprehensive understanding of parametric dependence of imperfection sensitivity on orthotropic material constants. Therefore, the present analysis is extended and described in terms of three bounded generic orthotropic constants (Kuo and Yang, 1988), namely, generalized rigidity ratio, D^*, generalized Poisson's ratio, ε, and principal rigidity ratio, α. They are defined as

$$D^* = \frac{Q_{12}+2Q_{33}}{\sqrt{Q_{11}Q_{22}}}, \quad \varepsilon = \frac{Q_{12}}{Q_{12}+2Q_{33}}, \quad \alpha = \sqrt{\frac{Q_{22}}{Q_{11}}} \tag{34}$$

and the bounds are

$$0 < D^* \leqslant 1$$
$$0.12 < \varepsilon < 0.65$$
$$0 < \alpha \leqslant 1.$$

The buckling loads N_c and N_r, in this formulation, are nondimensionalized in the following way:

$$\bar{N}_c(\bar{N}_r) = \frac{N_c(N_r)}{\sqrt{Q_{11}Q_{22}}}. \tag{35}$$

This treatment explores buckling behavior for the complete range of material parameters and furnishes very general conclusions in this regard. The force–displacement relations can be written in terms of these three global constants as described in Appendix B. The procedure for obtaining classical and reduced buckling loads remains exactly the same as described earlier. We can rewrite eqn (24) in generic terms by replacing A_{ij} by $\bar{A}_{ij}$ and using appropriate expressions for $\bar{A}_{ij}$ from Appendix B. The resulting expression for V_t/V_x is substituted back into eqn (23) such that the ratio of reduced to critical load, ρ, becomes

$$\rho = 1 - \left[\frac{\varepsilon D^* \lambda^2 + \alpha i^2}{\lambda^2(2 - \varepsilon^2 D^{*2}) + \alpha \varepsilon D^* i^2}\right] \varepsilon D^*. \tag{36}$$

This expression is valid for homogeneous orthotropic materials and $\theta = 0$.

Results and discussion

At first glance, the generic expression (36) shows that an increase in generalized Poisson's ratio, ε, or rigidity ratio, D^*, would cause a significant decrease in ρ. The generic constants for three common composites are reported in Table 3. The buckling loads are obtained for shell 3 of Table 2 and reported in Table 3. The comparison of ρ values shows that the glass–epoxy shell is more imperfection sensitive than those made of boron or graphite epoxy composites. This substantiates similar observations made by Khot (1970) and Semenyuk and Zhukova (1987). The parametric dependence of efficiency and imperfection sensitivity on generic constants is reported in Figs 4–7 for a shell geometry $L/R = 2$, $R/t = 250$ and $t = 0.016$.

It is observed in Fig. 4 that, with increasing D^* (α = const), ρ decreases and efficiency improves in monotonic fashion. When ε is increased from 0.2 to 0.6, the ρ and η curves drop comparatively but the nature of the variations remain qualitatively the same. The nature of the ρ vs D^* relation remains largely unchanged in Fig. 5. An increase in α from 0.6 to 0.9 improves efficiency but causes a relative drop in ρ. Figure 6 reveals that variations

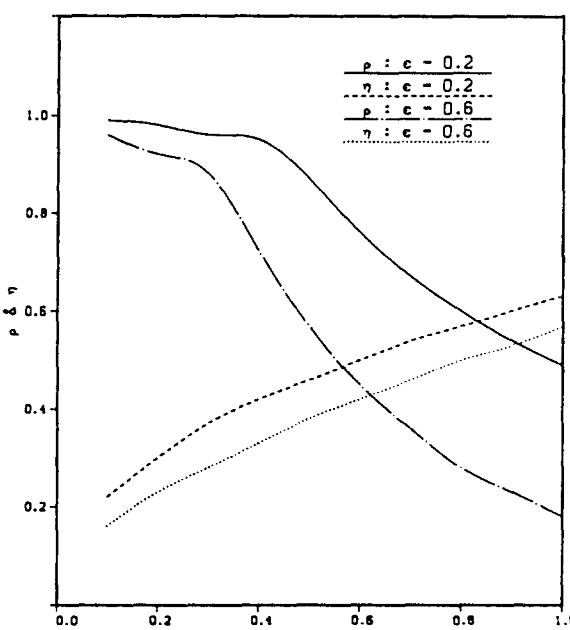

Fig. 4. ρ and η vs D^* plots ($\alpha = 0.3$).

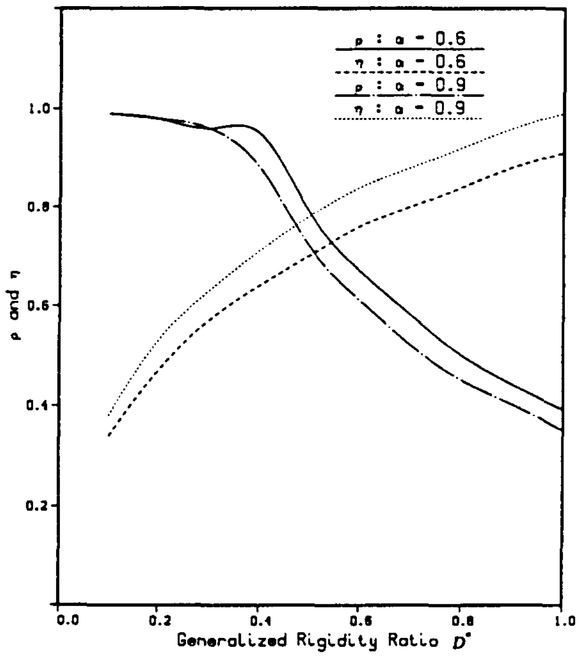

Fig. 5. ρ and η vs D^* plots ($\varepsilon = 0.2$).

in ρ with respect to α ($D^* = \text{const}$) are insignificant when compared to the efficiency which improves with increasing α. In Fig. 7, both ρ and η decrease with increasing ε. A relative increase in α improves efficiency significantly yet ρ remains largely unchanged for a wide range of ε.

The findings of generic analysis can be summarized as follows. The efficiency and imperfection sensitivity both increase with increasing D^*. The generalized Poisson's ratio has the worst effect as efficiency decreases and sensitivity increases with increases in its value. The principal rigidity ratio has the most benign effects; its increase improves efficiency significantly and imperfection sensitivity remains largely unchanged. Thus, a desirable

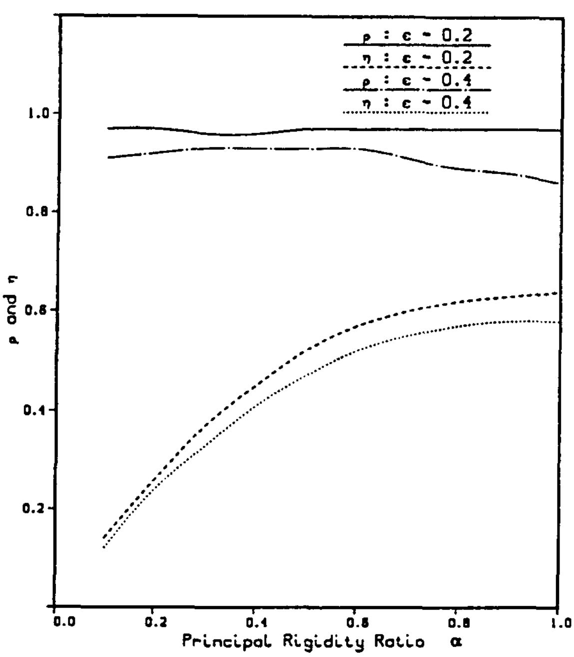

Fig. 6. ρ and η vs α plots ($D^* = 0.3$).

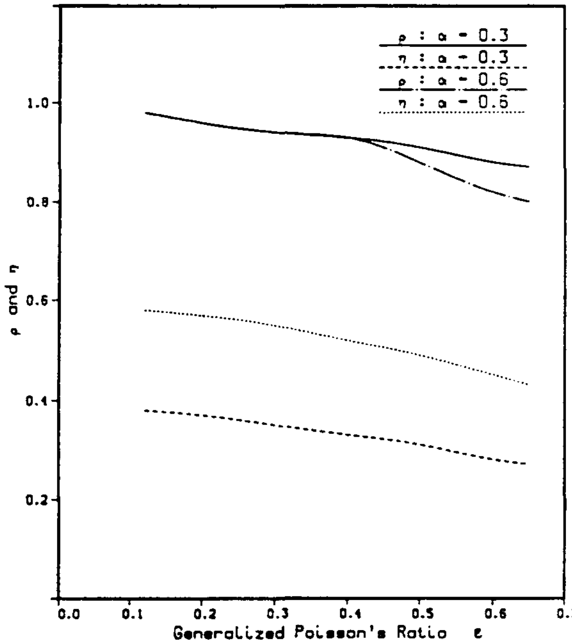

Fig. 7. ρ and η vs ε plots ($D^* = 0.3$).

laminate is one which has higher α, smaller D^* and smallest possible ε. Using the expressions listed in Appendix B, we can write

$$\bar{D}_{22} = \frac{\alpha}{12t^2}, \quad \bar{A}_{12} = \varepsilon D^*. \tag{37}$$

This immediately substantiates our conjecture that laminates with higher D_{22} (i.e. higher α) and lower A_{12} (i.e. lower ε or D^*) exhibit less imperfection sensitivity and further explains observations regarding relative imperfection sensitivity of multiple optimal laminates.

Semenyuk and Zhukova (1987) pointed out that Koiter's b-coefficient was related to the ratios (E_1/G) and (E_2/G), such that the larger these two ratios, the lower the imperfection sensitivity. It was suggested that optimizing the fiber orientation for the linear buckling load somehow decreases these ratios and, in turn, increases the imperfection sensitivity. The reduced stiffness approach provides an explanation for this. Substituting for Q_{ij} from the Notation section into eqn (36), the expression for D^* can be rewritten as

$$D^* = \frac{(\nu_{12}E_2/d)+2G}{(1/d)\sqrt{E_1E_2}} = \frac{(\nu_{12}/d)(E_2/G)+2}{(1/d)\sqrt{(E_1/G)(E_2/G)}}. \tag{38}$$

It can now be concluded that smaller values of D^* would be associated with higher values of (E_1/G) and (E_2/G) and would thereby exhibit less imperfection sensitivity. Figure 4 shows that optimization involving greater efficiency would require higher values of D^*, which is opposite to the requirement of reduced imperfection sensitivity. This clearly supports Semenyuk and Zhukova's (1987) results.

CONCLUSION

It is realized that highly sophisticated nonlinear, computerized analyses, available in the literature, are not able to provide a consistent physical model which might guide optimization procedures and, therefore, avoid undesirable postbuckling responses. The present paper is a comprehensive attempt to establish a rational theme using the reduced stiffness method. There is no intention of replacing existing work but rather of recognizing the physical characteristics present in advanced postbuckling and use them in an equivalent linear, eigenvalue analysis.

Some of the noteworthy conclusions are the following. The reduced stiffness model provides a relationship between laminate stiffness parameters' efficiency of buckling resistance and imperfection sensitivity in postbuckling deformation. Commonly, laminates with higher hoop bending stiffness and lower Poisson in-plane stiffness A_{12}, are expected to be less imperfection sensitive. In generic terms, laminates with higher principal rigidity ratio and lower generalized rigidity and Poisson ratios are favourable candidates for more stable postbuckling response. Optimization of the buckling load through a judicious choice of laminate configuration is often associated with increased imperfection sensitivity. It is observed that the criteria for optimality and reduced imperfection sensitivity are often opposed to each other, since an increase in laminate efficiency requires a corresponding increase in generalized rigidity ratio which, in turn, destabilizes the postbuckling behaviour.

The proposed physical approach successfully and consistently explains most of the observations reported in the literature which were based on nonlinear postbuckling analyses. This investigation recognizes a set of generic physical parameters and highlights their specific relationships with optimization and imperfection sensitivity. The reduced buckling load appears to be a useful indicator for evaluating qualitatively the relative imperfection sensitivity of various nearly optimal laminated shell designs at the conceptual rather than the final design stage.

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APPENDIX A: QUADRATIC POTENTIAL ENERGY

Strain energy

The strain energy, U_e, is the sum of its membrane (U_m), bending (U_b) and coupling (U_c) components, i.e.

$$U_e = U_m + U_b + U_c, \tag{A1}$$

where

$$U_m = \frac{C}{2R^2}[A_{11}\lambda^2U^2 - 2A_{12}\lambda U(W-iV) + A_{22}(W-iV)^2 + A_{33}(\lambda V+iU)^2]$$

$$U_b = \frac{CW^2}{2R^4}[D_{11}\lambda^4 + 2D_{12}i^2\lambda^2 + D_{22}i^4 + 4D_{33}i^2\lambda^2]$$

$$U_c = \frac{C}{R^3}[-B_{11}\lambda^3UW + B_{22}i^2W(W-iV)] \tag{A2}$$

and $C = R\pi L/2$.

Load potential terms

The load potential can be written as the algebraic sum of the work done in the axial and circumferential prebuckling deformations, V_x and V_t, respectively:

$$V_x = \frac{CW^2}{4R^2}[N_x\lambda^2 + (A_{11}\lambda^2 + A_{12}i^2)E_x]$$

$$V_t = \frac{CW^2}{4R^2}[A_{12}\lambda^2 + A_{22}i^2]E_t. \tag{A3}$$

APPENDIX B: STIFFNESS MATRICES IN GENERIC TERMS

The nondimensional stretching and bending stiffness matrices for a symmetric angle-ply can be obtained in the following manner (Kuo and Yang, 1988):

$$\bar{A}_{ij} = \frac{A_{ij}}{t\sqrt{Q_{11}Q_{22}}}$$

$$\bar{D}_{ij} = \frac{\bar{A}_{ij}}{12t^2}. \tag{B1}$$

The results are

$$\bar{A}_{11} = \left(\frac{1}{\alpha} - D^*\right)c^4 - (D^* - \alpha)s^4 + D^*$$

$$\bar{A}_{22} = \left(\frac{1}{\alpha} - D^*\right)s^4 - (D^* - \alpha)c^4 + D^*$$

$$\tilde{A}_{12} = \left(\alpha + \frac{1}{\alpha} - 2D^*\right)c^2s^2 + \varepsilon D^*$$

$$\tilde{A}_{33} = \left(\alpha + \frac{1}{\alpha} - 2D^*\right)c^2s^2 + \varepsilon D^* + \frac{D^*(1-\varepsilon)}{2} \quad \text{(B2)}$$

where

$$c = \cos\theta \quad \text{and} \quad s = \sin\theta.$$

The orthotropic invarients are defined as

$$\bar{U}_i = \frac{U_i}{\sqrt{Q_{11}Q_{22}}}, \quad i = 1,4,5. \quad \text{(B3)}$$

In terms of generic constants,

$$\bar{U}_1 = \frac{1}{8}\left(\frac{3}{\alpha} + 3\alpha + 2D^*\right)$$

$$\bar{U}_4 = \frac{1}{8}\left(\frac{1}{\alpha} + \alpha + 8\varepsilon D^* - 2D^*\right)$$

$$\bar{U}_5 = \frac{1}{8}\left(\frac{1}{\alpha} + \alpha - 4\varepsilon D^* + 2D^*\right). \quad \text{(B4)}$$

The buckling resistance of an equivalent isotropic laminate can be written in terms of the generic constants using eqns (30), (32) and (B4).